Asymptotic Expansions of the Lognormal Implied Volatility : A Model Free Approach

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Abstract

We invert the Black-Scholes formula. We consider the cases low strike, large strike, short maturity and large maturity. We give explicitly the first 5 terms of the expansions. A method to compute all the terms by induction is also given. At the money, we have a closed form formula for implied lognormal volatility in terms of a power series in call price.

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1 Introduction

1.1 Overview

In a market with no arbitrage, the price of a call option can take two extreme values : its "intrinsic value" which is equal to the payoff of the option (lower boundary value) and the spot price (upper boundary value). For simplicity, we assume a market with no interest rate. Otherwise, we would consider the forward price. We will consider here the case when the call price is close to its boundary value and we will obtain in that case an approximation of the corresponding lognormal implied volatility. This case happens in particular when the maturity of the option is small. To be precise, in the case when $T \to 0$ (resp. $T \to +\infty$), we will obtain an asymptotic expansion of the implied lognormal volatility as a sum of terms of the form $\lambda^i \ln^j(\lambda)$ with $j < i$ and $\lambda = -\frac{1}{\ln\left(\frac{C(T,K)-(S-K)_+}{S}\right)}$ (resp. $\lambda = -\frac{1}{\ln\left(\frac{S-C(T,K)}{S}\right)}$) where $C(T,K)$ denotes the price of a call option with strike K, maturity T and spot price S (Proposition 5). Note that here, the spot price S is present only to insure that the ratio $\frac{C(T,K)-(S-K)_+}{S}$ (resp. $\frac{S-C(T,K)}{S}$)is with no-dimension. The important quantity is the "time-value" $\mathrm{TV}(T,K) := C(T,K) - (S-K)_+$ (resp. "covered call" $\mathrm{CC}(\mathrm{T},\mathrm{K}) := S - C(T,K)$) The computations involve no complicated formulas except may be a well known

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asymptotic expansion for the incomplete Gamma function (Equation (16)). The interest of such a formula is twofold. First, it gives quickly an easy approximation of the true implied lognormal volatility. This can serve as a starting point for the calculus of the exact implied lognormal volatility using a Newton method for instance. The formula can also be useful to transform theoretical approximations of a call price into approximations of implied lognormal volatility. Indeed, asymptotics of call prices can be obtained with the help of stochastic differential equations of partial differential equations using perturbation methods. Then, a transformation has to be made to obtain the implied lognormal volatility which is of a fundamental interest for the practitioner.

All our work is based on a single inversion formula. Explicitly we invert the following equation for $\lambda \ll 1$ and $\beta > 0$ (see Note 2):

$$v^{\beta} \mathrm{e}^{-\frac{1}{v}} \left[\sum_{k=0}^{N} \alpha_k \, v^k \; + O\left(v^{N+1}\right) \right] = \mathrm{e}^{\gamma} \, \mathrm{e}^{-\frac{1}{\lambda}} \tag{1}$$

The good framework for solving this problem (i.e. obtain v in terms of λ) is the theory of transseries (see [6]). In the expansion of v in terms of λ coming from (1) it is important to go up to order 5 (for us, order 0 is λ, order 1 is $\lambda^2 \ln(\lambda)$, ... order 5 is λ^3) to see α_1 (see Lemma 1):

$$v = \lambda - \beta\lambda^2 \ln\lambda + \gamma\lambda^2 + \beta^2\lambda^3 \ln^2(\lambda) + \left(\beta^2 - 2\beta\gamma\right) \lambda^3 \ln(\lambda) + \left(\gamma^2 - \beta\gamma - \alpha_1\right) \lambda^3 + o\left(\lambda^3\right)$$

1.2 Basic definitions

In a Black-Scholes world, the dynamic of a stock (S_t) is given by:

$$dS_t = \sigma_{LN} S_t dW_t,$$

with initial value S at $t = 0$. The so-called lognormal volatility σ_{LN} is related to the price of a call $\mathrm{BS}\left(S, K, T, \sigma\right)$ struck at K with maturity T by the Black-Scholes formula (See [2]):

$$\mathrm{BS}\left(S, K, T, \sigma\right) = SN\left(d_+\right) - KN\left(d_-\right) \tag{2}$$

with

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du$$

and

$$d_{\pm} = \frac{\ln\left(\dfrac{S}{K}\right) \pm \dfrac{\sigma_{LN}^2 T}{2}}{\sigma_{LN}\sqrt{T}}$$

To simplify matters, we have considered $r = 0$. Otherwise, we would consider the forward price $F_t = S_t \mathrm{e}^{rt}$ instead of the spot price S_t. Following Ropper-Rutkowski ([4]), we set:

Definition 1 *Let us denote by:*

- $\mathrm{TV}(S, K, K, T)$ *(or simply* TV(T, K) *or* TV*) the time-value of a European call option struck at strike* K *with maturity* T*:* $\mathrm{TV}\left(S, K, T, \sigma\right) := \mathrm{BS}\left(S, K, T, \sigma\right) - \left(S - K\right)_+$
- $x := \ln(\frac{K}{S})$ *(the log-moneyness)*

- $\theta := \sigma_{LN}\sqrt{T}$ *(the square root of the time-variance)*

The spot price S is assumed to be fixed by the market. We will consider the two following cases: K is fixed and $\sigma\sqrt{T}$ is small (case 1) and $\sigma\sqrt{T}$ is fixed and $\frac{K}{S}$ is large (case 2). In both cases, we will obtain a similar expression for the asymptotic expansion of the implied lognormal volatility.

2 Asymptotic expansions of a European call option

First let us assume that $x \neq 0$.

2.1 Asymptotic expansions of a European call option for $x \neq 0$.

We note that the expression giving the time-value of a call-option in the case ($\theta \ll 1$ and x fixed) is very similar to the case ($|x| \gg 1$ and θ fixed).

Proposition 1 *(Case 1.) Let $N \in \mathbb{N}$. When $\theta \to 0$ and x fixed, the asymptotic expansion of the time-value $TV = C(T,K) - (S-K)_+$ of a call price is given at order N by:*

$$4\sqrt{\pi}\,\frac{\mathrm{e}^{-\frac{x}{2}}}{|x|}\left(\frac{TV}{S}\right) = \left(\frac{2\theta^2}{x^2}\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{1}{\left(\frac{2\theta^2}{x^2}\right)}} \sum_{k=0}^{N} \frac{(-1)^k}{2^k}\, a_k\left(\frac{x^2}{8}\right)\left(\frac{2\theta^2}{x^2}\right)^k + O\left(\theta^{2N+5}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\right) \quad (3)$$

with

$$a_k(z) := (2k+1)!!\, f_k(z) \quad (4)$$

$$f_k(z) := \sum_{j=0}^{k} \frac{z^j}{j!\,(2j+1)!!} \quad (5)$$

and for $j \in \mathbb{Z}$, $(2j+1)!! := \prod_{l=1}^{j}(2l+1)$ (with the convention $\prod_{\emptyset} := 1$).

(Case 2.) Let $N \in \mathbb{N}$. When $|x| \to +\infty$ (i.e., $K \to 0$ or $K \to +\infty$) and θ fixed, the asymptotic expansion of the time-value of a call price is given at order N by:

$$2\sqrt{2\pi}\,\frac{\mathrm{e}^{-\frac{x}{2}}}{\theta}\left(\frac{TV}{S}\right) = \left(\frac{2\theta^2}{x^2}\right) \mathrm{e}^{-\frac{1}{\left(\frac{2\theta^2}{x^2}\right)}} \sum_{k=0}^{N} \frac{(-1)^k}{2^k}\, b_k\left(\frac{\theta^2}{4}\right)\left(\frac{2\theta^2}{x^2}\right)^{2k} + O\left(x^{-2N-4}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\right) \quad (6)$$

with

$$b_k(z) := (2k+1)!! \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{z^j}{(2j+1)!!} \quad (7)$$

.

(Case 3.) Let $N \in \mathbb{N}$. When $\theta \to +\infty$ and x fixed, the asymptotic expansion of the covered call $CC = S - C(T,K)$ of a call price is given at order N by:

$$\sqrt{\pi}\,\mathrm{e}^{-\frac{x}{2}}\,\frac{\mathrm{CC}}{S} = \left(\frac{8}{\theta^2}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{\left(\frac{8}{\theta^2}\right)}} \sum_{k=0}^{N} \frac{(-1)^k}{2^k}\, c_k\left(\frac{x^2}{8}\right)\left(\frac{8}{\theta^2}\right)^k + O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \quad (8)$$

with

$$c_k(z) := (2k-1)!!\, g_k(z) \tag{9}$$

$$g_k(z) := \sum_{j=0}^{k} \frac{z^j}{j!\,(2j-1)!!} \tag{10}$$

Note that $g'_k(z) = f_{k-1}(z)$ and $c'_k(z) = a_{k-1}(z)$.

Proof. *Case* 1. For $n \in \mathbb{N}$, we denote by $\tilde{e}_n$ the function defined by

$$\forall\, u \in \mathbb{R}, \quad \mathrm{e}^{-u} = 1 - u + \frac{u^2}{2!} - ... + (-1)^n \frac{u^n}{n!} + \tilde{e}_n(u) \tag{11}$$

Then, it is classical (properties of alternate series) that

$$\forall\, u > 0, \quad |\tilde{e}_n(u)| \le \frac{u^{n+1}}{(n+1)!}. \tag{12}$$

Now, let us fix $N \in \mathbb{N}$. We start from:

$$\sqrt{2\pi}\, \mathrm{e}^{-\frac{x}{2}}\, \frac{TV}{S} = \int_0^\theta \mathrm{e}^{-\frac{1}{2}\left(\frac{x^2}{\xi^2} + \frac{\xi^2}{4}\right)} d\xi \tag{13}$$

with $x = \ln\left(\frac{K}{S}\right)$ as before. This formula can be obtained by deriving the Black-Scholes formula with respect to θ and then integrating the result (See [RR], Lemma 3.1). We have:

$$\sqrt{2\pi}\, \mathrm{e}^{-\frac{x}{2}}\, \frac{TV}{S} = \int_0^\theta \mathrm{e}^{-\frac{x^2}{2\xi^2}}\, \mathrm{e}^{-\frac{\xi^2}{8}}\, d\xi \tag{14}$$

So, by (11),

$$\sqrt{2\pi}\, \mathrm{e}^{-\frac{x}{2}}\, \frac{TV}{S} = \sum_{n=0}^{N} \frac{(-1)^n}{n!\, 8^n} \int_0^\theta \mathrm{e}^{-\frac{x^2}{2\xi^2}}\, \xi^{2n} d\xi + \int_0^\theta \mathrm{e}^{-\frac{x^2}{2\xi^2}}\, \tilde{e}_N\left(\frac{\xi^2}{8}\right) d\xi$$

So, with the change of variables $u := \frac{x^2}{2\xi^2}$, we get:

$$\begin{aligned}
\sqrt{2\pi}\, \mathrm{e}^{-\frac{x}{2}}\, \frac{TV}{S} &= \sum_{n=0}^{N} \frac{(-1)^n}{n!\, 16^n}\, \frac{|x|^{2n+1}}{2\sqrt{2}} \int_{\frac{x^2}{2\theta^2}}^{+\infty} u^{-n-\frac{3}{2}} \mathrm{e}^{-u}\, du + R_N(\theta) \\
&= \sum_{n=0}^{N} \frac{(-1)^n}{n!\, 16^n}\, \frac{|x|^{2n+1}}{2\sqrt{2}}\, \Gamma\left(-n - \frac{1}{2}, \frac{x^2}{2\theta^2}\right) + R_N(\theta)
\end{aligned} \tag{15}$$

with $R_N(\theta) := \int_0^\theta \mathrm{e}^{-\frac{x^2}{2\xi^2}}\, \tilde{e}_n\left(\frac{\xi^2}{8}\right) d\xi$.
We have:

$$
\begin{aligned}
|R_N(\theta)| &\le \int_0^\theta e^{-\frac{x^2}{2\xi^2}} \, |\tilde{e}_N\left(\frac{\xi^2}{8}\right)| \, d\xi \\
&\le \int_0^\theta e^{-\frac{x^2}{2\xi^2}} \left(\frac{1}{(N+1)!\,8^{N+1}}\right) \xi^{2(N+1)} d\xi \\
&\le \frac{1}{(N+1)!\,8^{N+1}} \int_0^\theta \xi^{2(N+1)} \, e^{-\frac{x^2}{2\xi^2}} \, d\xi \\
&\le \frac{1}{(N+1)!\,8^{N+1}} \int_{\frac{x^2}{2\theta^2}}^{+\infty} \left(\frac{x^2}{2}\right)^{N+1} \frac{|x|}{2\sqrt{2}} \, u^{-(N+1)-\frac{3}{2}} \, e^{-u} du \\
&\le \frac{|x|^{2N+3}}{2\sqrt{2}\,(N+1)!\,16^{N+1}} \, \Gamma\left(-N-\frac{3}{2}, \frac{x^2}{2\theta^2}\right)
\end{aligned}
$$

We recall the following asymptotic expansion valid for $z \to +\infty$ and $m \in \mathbb{N}$ (see Formula 6.5.32 in ([1])):

$$
\Gamma(a,z) = z^{a-1} \, e^{-z} \left[1 + \frac{a-1}{z} + ... + \frac{(a-1)...(a-m)}{z^m}\right] + \gamma_m(a,z) \tag{16}
$$

with

$$
\gamma_m(a,z) = O\left(\frac{z^{a-1}\, e^{-z}}{z^{m+1}}\right) \tag{17}
$$

In particular, with $a = -N - \frac{3}{2}$, $z = \frac{x^2}{2\theta^2}$ and $m = 0$, in the limit when $\theta \to 0$, we get:

$$
\Gamma\left(-N-\frac{3}{2}, \frac{x^2}{2\theta^2}\right) = O\left[\left(\frac{x^2}{2\theta^2}\right)^{-N-\frac{5}{2}} e^{-\frac{x^2}{2\theta^2}}\right].
$$

So, when $\theta \to 0$,

$$
R_N(\theta) = O\left(\theta^{2N+5} \, e^{-\frac{x^2}{2\theta^2}}\right) \tag{18}
$$

Moreover, for any $n < N$, we have by (16) with $m = N - n$, $a = -n - \frac{1}{2}$, $z = \frac{x^2}{2\theta^2}$:

$$\begin{aligned}
\Gamma\left(-n-\frac{1}{2},\frac{x^2}{2\theta^2}\right) &= \left(\frac{x^2}{2\theta^2}\right)^{-n-\frac{3}{2}} \mathrm{e}^{-\frac{x^2}{2\theta^2}} \times \\
&\times \left[1+\frac{\left(-n-\frac{1}{2}-1\right)}{\left(\frac{x^2}{2\theta^2}\right)^{1}}+\ldots+\frac{\left(-n-\frac{1}{2}-1\right)\ldots\left(-n-\frac{1}{2}-(N-n)\right)}{\left(\frac{x^2}{2\theta^2}\right)^{N-n}}\right]+ \\
&+ \gamma_{N-n}\left(-n-\frac{1}{2},\frac{x^2}{2\theta^2}\right) \\
&= \left(\frac{2\theta^2}{x^2}\right)^{\frac{3}{2}} \times \mathrm{e}^{-\frac{x^2}{2\theta^2}} \\
&\times \left[\left(\frac{2\theta^2}{x^2}\right)^{n}+\frac{(-1)}{2}\frac{(2n+3)!!}{(2n+1)!!}\left(\frac{2\theta^2}{x^2}\right)^{n+1}+\ldots+\frac{(-1)^{N-n}}{2^{N-n}}\frac{(2N+1)!!}{(2n+1)!!}\left(\frac{2\theta^2}{x^2}\right)^{N}\right] \\
&+ \gamma_{N-n}\left(-n-\frac{1}{2},\frac{x^2}{2\theta^2}\right) \\
&= \left(\frac{2\theta^2}{x^2}\right)^{\frac{3}{2}} \times \mathrm{e}^{-\frac{x^2}{2\theta^2}} \left[\sum_{k=n}^{N}\frac{(-1)^{k-n}}{2^{k-n}}\frac{(2k+1)!!}{(2n+1)!!}\left(\frac{2\theta^2}{x^2}\right)^{k}\right] \\
&+ \gamma_{N-n}\left(-n-\frac{1}{2},\frac{x^2}{2\theta^2}\right) \qquad (19)
\end{aligned}$$

Moreover, when $\theta \to 0$, we have by (17):

$$\begin{aligned}
\gamma_{N-n}\left(-n-\frac{1}{2},\frac{x^2}{2\theta^2}\right) &= O\left(\frac{\left(\frac{x^2}{2\theta^2}\right)^{-n-\frac{3}{2}} \mathrm{e}^{-\frac{x^2}{2\theta^2}}}{\left(\frac{x^2}{2\theta^2}\right)^{N-n+1}}\right) \\
&= O\left(\theta^{2N+5}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\right) \qquad (20)
\end{aligned}$$

Therefore, by (15), (18), (19), (20), we obtain:

$$\begin{aligned}
\sqrt{2\pi}\,\frac{\mathrm{e}^{-\frac{x}{2}}}{|x|}\,\frac{TV}{S} &= \left(\frac{2\theta^2}{x^2}\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{x^2}{2\theta^2}} \sum_{n=0}^{N}\frac{(-1)^n}{n!\,16^n}\frac{|x|^{2n}}{2\sqrt{2}}\sum_{k=n}^{N}\frac{(-1)^{k-n}}{2^{k-n}}\frac{(2k+1)!!}{(2n+1)!!}\left(\frac{2\theta^2}{x^2}\right)^{k} \\
&+ O\left(\theta^{2N+5}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\right)
\end{aligned}$$

Hence,

$$4\sqrt{\pi}\,\frac{\mathrm{e}^{-\frac{x}{2}}}{|x|}\left(\frac{TV}{S}\right) = \left(\frac{2\theta^2}{x^2}\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{x^2}{2\theta^2}} \sum_{k=0}^{N}\frac{(-1)^k}{2^k}\,a_k\left(\frac{x^2}{8}\right)\left(\frac{2\theta^2}{x^2}\right)^{k} + O\left(\theta^{2N+5}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\right)$$

with

$$a_k\left(\frac{x^2}{8}\right) := (2k+1)!!\sum_{j=0}^{k}\frac{1}{j!\,(2j+1)!!}\left(\frac{x^2}{8}\right)^j$$

which is exactly Proposition 1 - (3).

Case 2: θ is fixed and $|x| \to +\infty$. Set:

$$I(x) := \int_0^\theta \mathrm{e}^{-\frac{x^2}{2\xi^2}}\,\mathrm{e}^{-\frac{\xi^2}{8}}\,d\xi. \tag{21}$$

With the help of the change of variables $z = \dfrac{\theta^2}{\xi^2}$, we have:

$$I(x) = \frac{\theta}{2}\int_1^{+\infty}\mathrm{e}^{-\frac{x^2}{2\theta^2}z}\,\tilde{h}(z)\,dz$$

with $\tilde{h}(z) := \dfrac{16\sqrt{2}}{\theta^3}\,f_{-\frac{3}{2}}\left(\dfrac{8z}{\theta^2}\right)$ and $f_\alpha(z) := z^\alpha\,\mathrm{e}^{-\frac{1}{z}}$. By induction on n, we show that

$$\forall n\in\mathbb{N},\ \forall z\in\mathbb{R}_+^*,\quad f_\alpha^{(n)}(z) = z^{\alpha-2n}\mathrm{e}^{-\frac{1}{z}}\sum_{p=0}^{n}\binom{n}{p}[\alpha-n+p]_p\,z^p \tag{22}$$

where $f_\alpha^{(n)}$ is the n^{th} derivative of f_α and with by definition, $[u]_k := \prod_{j=0}^{k-1}(u-j)$ for any real u and integer k. In particular, for any $(\alpha, N)\in\mathbb{R}_-\times\mathbb{N}^*$ fixed, $f_\alpha^{(N)}(z)$ is uniformly bounded in $z\in\mathbb{R}_+^*$. Therefore, $\tilde{h}^{(N)}(z)$ is also uniformly bounded in $z\in\mathbb{R}_+^*$. So $h^{(N)}(z)$ is also uniformly bounded in $z\in\mathbb{R}_+^*$ with $h(z) := \tilde{h}(z+1)$ (the function h is analytic on $\mathbb{R}_+$), i.e.,

$$\forall N\in\mathbb{N}\,\exists\, M_N\in\mathbb{R}_+\,\forall z\in\mathbb{R}_+,\quad h^{(N)}(z)\le M_N \tag{23}$$

Let us fix $N\in\mathbb{N}$. By Taylor-Lagrange, we get:

$$\begin{aligned} I(x) &= \frac{\theta}{2}\int_0^{+\infty}\mathrm{e}^{-\frac{x^2}{2\theta^2}(z+1)}h(z)\,dz \\ &= \frac{\theta}{2}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\int_0^{+\infty}\mathrm{e}^{-\frac{x^2}{2\theta^2}z}\left(\sum_{k=0}^{N}\frac{h^{(k)}(0)}{k!}z^k + R_{N+1}(z)\right)dz \end{aligned}$$

with $R_{N+1}(z)\le M_{N+1}\dfrac{z^{N+1}}{(N+1)!}$. So,

$$I(x) = \frac{\theta}{2}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\sum_{k=0}^{N}\frac{h^{(k)}(0)}{k!}\int_0^{+\infty}\mathrm{e}^{-\frac{x^2}{2\theta^2}z}z^k\,dz + \frac{\theta}{2}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\int_0^{+\infty}\mathrm{e}^{-\frac{x^2}{2\theta^2}z}R_{N+1}(z)\,dz$$

Using the fact that

$$\forall A\in\mathbb{R}_+^*,\quad \int_0^{+\infty}\mathrm{e}^{-Az}z^n\,dz = \frac{n!}{A^{n+1}} \tag{24}$$

we get:

$$
\begin{aligned}
I(x) &= \frac{\theta}{2}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}} \sum_{k=0}^{N} \left[\frac{h^{(k)}(0)}{k!}\,k!\,\left(\frac{2\theta^2}{x^2}\right)^{k+1}\right] + O\left(\mathrm{e}^{-\frac{x^2}{2\theta^2}}\left(\frac{\theta^2}{x^2}\right)^{N+2}\right)\\
&= \frac{\theta}{2}\left(\frac{2\theta^2}{x^2}\right)\mathrm{e}^{-\frac{x^2}{2\theta^2}} \sum_{k=0}^{N} \tilde{h}^{(k)}(1)\left(\frac{2\theta^2}{x^2}\right)^{k} + O\left(\mathrm{e}^{-\frac{x^2}{2\theta^2}}\left(\frac{\theta^2}{x^2}\right)^{N+2}\right)\\
&= \frac{\theta}{2}\left(\frac{2\theta^2}{x^2}\right)\mathrm{e}^{-\frac{x^2}{2\theta^2}} \sum_{k=0}^{N} \tilde{h}^{(k)}(1)\left(\frac{2\theta^2}{x^2}\right)^{k} + O\left(x^{-2N-4}\,\mathrm{e}^{-\frac{x^2}{2\theta^2}}\right) \qquad (25)
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{h}^{(k)}(1) &:= \frac{16\sqrt{2}}{\theta^3}\left(\frac{d^k}{z^k}\left[f_{-\frac{3}{2}}\,\frac{8z}{\theta^2}\right]\right)_{z=1}\\
&= \frac{16\sqrt{2}}{\theta^3}\left(\frac{8}{\theta^2}\right)^{k} f^{(k)}_{-\frac{3}{2}}\left(\frac{8}{\theta^2}\right)\\
&= \frac{16\sqrt{2}}{\theta^3}\left(\frac{8}{\theta^2}\right)^{k}\left(\frac{8}{\theta^2}\right)^{-\frac{3}{2}-2k}\sum_{j=0}^{k}\binom{k}{j}\left[-\frac{3}{2}-k+j\right]_j\left(\frac{8}{\theta^2}\right)^{j}\\
&= \left(\frac{\theta^2}{8}\right)^{k}\sum_{j=0}^{k}\binom{k}{j}\left[-\frac{3}{2}-k+j\right]_j\left(\frac{\theta^2}{8}\right)^{-j}\\
&= \sum_{j=0}^{k}\binom{k}{j}\left[-\frac{3}{2}-(k-j)\right]_j\left(\frac{\theta^2}{8}\right)^{k-j}\\
&= \sum_{j=0}^{k}\binom{k}{j}\left[-\frac{3}{2}-j\right]_{k-j}\left(\frac{\theta^2}{8}\right)^{j}\\
&= \sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}}{2^{k-j}}\frac{(2k+1)!!}{(2j+1)!!}\left(\frac{\theta^2}{8}\right)^{j}\\
&= \frac{(-1)^k}{2^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(2k+1)!!}{(2j+1)!!}\left(\frac{\theta^2}{4}\right)^{j} \qquad (26)
\end{aligned}
$$

Using (14), (21), (25) and (26), this concludes the proof of Case 2.

Case 3. We know turn to the case $\theta \to +\infty$. We start again from

$$
\sqrt{2\pi}\,\mathrm{e}^{-\frac{x}{2}}\,\frac{C(T,K)-(S-K)_+}{S} = \int_0^{\theta} \mathrm{e}^{-\frac{1}{2}\left(\frac{x^2}{\xi^2}+\frac{\xi^2}{4}\right)} d\xi
$$

When $\theta \to +\infty$, $C(T,K) \to S$. So,

$$
\sqrt{2\pi}\,\mathrm{e}^{-\frac{x}{2}}\,\frac{\mathrm{CC}}{S} = \int_{\theta}^{+\infty} \mathrm{e}^{-\frac{1}{2}\left(\frac{x^2}{\xi^2}+\frac{\xi^2}{4}\right)} d\xi
$$

with $\mathrm{CC} := S - C(T,K)$. By the change of variables $\eta := \dfrac{\xi^2}{8}$, we get:

$$\sqrt{2\pi}\,\mathrm{e}^{-\frac{x}{2}}\,\frac{\mathrm{CC}}{S} = \int_{\frac{\theta^2}{8}}^{+\infty} \mathrm{e}^{-\frac{x^2}{16\,\eta}}\,\mathrm{e}^{-\eta}\left(\sqrt{2}\eta^{-\frac{1}{2}}d\eta\right)$$

So, with the notations of (11) and with $N \in \mathbb{N}^*$,

$$\begin{aligned}
\sqrt{\pi}\,\mathrm{e}^{-\frac{x}{2}}\,\frac{\mathrm{CC}}{S} &= \int_{\frac{\theta^2}{8}}^{+\infty} \mathrm{e}^{-\frac{x^2}{16}}\,\eta^{-\frac{1}{2}}\,\mathrm{e}^{-\eta}d\eta \\
&= \int_{\frac{\theta^2}{8}}^{+\infty}\left[\sum_{k=0}^{N}\frac{(-1)^k}{k!}\left(\frac{x^2}{16\eta}\right)^k + \tilde{e}_N\left(\frac{x^2}{16\,\eta}\right)\right]\eta^{-\frac{1}{2}}\,\mathrm{e}^{-\eta}d\eta \\
&= \sum_{k=0}^{N}\frac{(-1)^k}{k!}\left(\frac{x^2}{16}\right)^k\int_{\frac{\theta^2}{8}}^{+\infty}\eta^{-k-\frac{1}{2}}\,\mathrm{e}^{-\eta}d\eta + \int_{\frac{\theta^2}{8}}^{+\infty}\tilde{e}_N\left(\frac{x^2}{16\,\eta}\right)\eta^{-\frac{1}{2}}\,\mathrm{e}^{-\eta}d\eta \\
&= \sum_{k=0}^{N}\frac{(-1)^k}{k!}\left(\frac{x^2}{16}\right)^k\Gamma\left(-k+\frac{1}{2},\frac{\theta^2}{8}\right) + \int_{\frac{\theta^2}{8}}^{+\infty}\tilde{e}_N\left(\frac{x^2}{16\,\eta}\right)\eta^{-\frac{1}{2}}\,\mathrm{e}^{-\eta}d\eta \qquad (27)
\end{aligned}$$

Moreover, for $\theta \to +\infty$, we have by (12) and (17):

$$\begin{aligned}
\left|\int_{\frac{\theta^2}{8}}^{+\infty}\tilde{e}_N\left(\frac{x^2}{16\,\eta}\right)\eta^{-\frac{1}{2}}\,\mathrm{e}^{-\eta}d\eta\right| &\leq \int_{\frac{\theta^2}{8}}^{+\infty}\frac{1}{(N+1)!}\left(\frac{x^2}{16\,\eta}\right)^{N+1}\eta^{-\frac{1}{2}}\,\mathrm{e}^{-\eta}d\eta \\
&\leq \frac{1}{(N+1)!}\left(\frac{x^2}{16}\right)^{N+1}\Gamma\left(-N-\frac{1}{2},\frac{\theta^2}{8}\right) \\
&= O\left(\left(\frac{\theta^2}{8}\right)^{-N-\frac{3}{2}}\mathrm{e}^{-\frac{\theta^2}{8}}\right) \\
&= O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \qquad (28)
\end{aligned}$$

On the other hand, with the notations of (16) and $N, k \in \mathbb{N}$ with $N \geq k$,

$$\begin{aligned}
\Gamma\left(-k+\frac{1}{2},\frac{\theta^2}{8}\right) &= \left(\frac{\theta^2}{8}\right)^{-k-\frac{1}{2}}\mathrm{e}^{-\frac{\theta^2}{8}}\left[1+\frac{-k-\frac{1}{2}}{\frac{\theta^2}{8}}+\ldots+\frac{(-k-\frac{1}{2})\ldots(-k-\frac{1}{2}-(N-k)+1)}{\left(\frac{\theta^2}{8}\right)^{N-k}}\right] \\
&+ \gamma_{N-k}\left(-k+\frac{1}{2},\frac{\theta^2}{8}\right) \\
&= \left(\frac{\theta^2}{8}\right)^{-\frac{1}{2}}\mathrm{e}^{-\frac{\theta^2}{8}}\left[\sum_{j=k}^{N}\frac{\prod_{l=1}^{j-k}(-k+\frac{1}{2}-l)}{\left(\frac{\theta^2}{8}\right)^{j}}\right] + \gamma_{N-k}\left(-k+\frac{1}{2},\frac{\theta^2}{8}\right) \qquad (29)
\end{aligned}$$

with

$$\begin{aligned}
\left|\gamma_{N-k}\left(-k+\frac{1}{2},\frac{\theta^2}{8}\right)\right| &\leq \frac{(-k-\frac{1}{2})\ldots(-k-\frac{1}{2}-(N-k))}{\left(\frac{\theta^2}{8}\right)^{N-k+1}}\,\mathrm{e}^{-\frac{\theta^2}{8}} \\
&= O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \qquad (30)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\sqrt{\pi}\,\mathrm{e}^{-\frac{x}{2}}\,\frac{CC}{S} &= \sum_{k=0}^{N}\frac{(-1)^k}{k!}\left(\frac{x^2}{16}\right)^k\left(\frac{\theta^2}{8}\right)^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{\theta^2}{8}}\left[\sum_{j=k}^{N}\frac{\prod_{l=1}^{j-k}(-k+\frac{1}{2}-l)}{\left(\frac{\theta^2}{8}\right)^j}\right] \\
&+\; O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \\
&= \left(\frac{8}{\theta^2}\right)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{\theta^2}{8}}\sum_{k=0}^{N}\frac{(-1)^k}{k!}\left(\frac{x^2}{16}\right)^k\left[\sum_{j=k}^{N}\frac{\prod_{l=1}^{j-k}(-k+\frac{1}{2}-l)}{\left(\frac{\theta^2}{8}\right)^j}\right] \\
&+\; O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \\
&= \left(\frac{8}{\theta^2}\right)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{\theta^2}{8}}\sum_{j=0}^{N}\left(\frac{8}{\theta^2}\right)^j\sum_{k=0}^{j}\frac{(-1)^k}{k!}\left(\frac{x^2}{16}\right)^k\prod_{l=1}^{j-k}(-k+\frac{1}{2}-l) \\
&+\; O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \qquad (31)
\end{aligned}
$$

We have

$$
\prod_{l=1}^{j-k}(-k+\frac{1}{2}-l) = \frac{(-1)^{j-k}}{2^{j-k}}\frac{(2j-1)!!}{(2k-1)!!}
$$

Hence,

$$
\begin{aligned}
\sqrt{\pi}\,\mathrm{e}^{-\frac{x}{2}}\,\frac{\mathrm{CC}}{S} &= \left(\frac{8}{\theta^2}\right)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{\left(\frac{8}{\theta^2}\right)}}\sum_{j=0}^{N}\left(\frac{8}{\theta^2}\right)^j\sum_{k=0}^{j}\frac{(-1)^k}{k!}\left(\frac{x^2}{16}\right)^k\frac{(-1)^{j-k}}{2^{j-k}}\frac{(2j-1)!!}{(2k-1)!!} \\
&+\; O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \qquad (32) \\
&= \left(\frac{8}{\theta^2}\right)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{\left(\frac{8}{\theta^2}\right)}}\sum_{k=0}^{N}\frac{(-1)^k}{2^k}\;c_k\left(\frac{x^2}{8}\right),\left(\frac{8}{\theta^2}\right)^k + O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \qquad (33)
\end{aligned}
$$

with

$$
c_k\left(\frac{x^2}{8}\right) = (2k-1)!!\sum_{j=0}^{k}\frac{1}{j!\,(2j-1)!!}\left(\frac{x^2}{8}\right)^j
$$

This is exactly (8) and it puts an end to the proof of Proposition 1.

2.2 Asymptotic expansions of a European call option for $x = 0$.

At the money, we have $(S-K)_+ = 0$. So, TV $= C$ and by (14):

$$
\sqrt{2\pi}\,\frac{C}{S} = \int_0^{\theta}\mathrm{e}^{-\frac{\xi^2}{8}}\,d\xi
$$

So,

Proposition 2 *At the money,*

$$C \;=\; S\,\mathrm{erf}\left(\frac{\theta}{2\sqrt{2}}\right) \tag{34}$$

with $\mathrm{erf}(u) := \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^u \mathrm{e}^{-\zeta^2}\,d\zeta.$

In the same way, we have:

$$\frac{CC}{S} \;=\; \mathrm{erfc}\left(\frac{\theta}{2\sqrt{2}}\right) \tag{35}$$

with $\mathrm{erfc}(u) := \dfrac{2}{\sqrt{\pi}}\displaystyle\int_u^{+\infty} \mathrm{e}^{-\zeta^2}\,d\zeta.$

Proposition 3 *Let $N \in \mathbb{N}$. (Case 1.) For $\theta \to 0$ and $\theta \neq 0$, we have:*

$$\sqrt{2\pi}\,\frac{C}{S} = \theta\sum_{k=0}^{N}\frac{(-1)^k}{2^k}\cdot\frac{1}{(2k+1)\,k!}\left(\frac{\theta^2}{4}\right)^k + O\left(\theta^{2N+3}\right) \tag{36}$$

(Case 2.) For $\theta \to +\infty$, we have:

$$\sqrt{\pi}\,\frac{CC}{S} = \left(\frac{8}{\theta^2}\right)^{\frac{1}{2}}\mathrm{e}^{-\frac{\theta^2}{8}}\sum_{k=0}^{N}\frac{(-1)^k}{2^k}.(2k-1)!!\left(\frac{8}{\theta^2}\right)^k + O\left(\theta^{-2N-3}\,\mathrm{e}^{-\frac{\theta^2}{8}}\right) \tag{37}$$

Proof. Formula (37) comes from the well known asymptotic expansion of $\mathrm{erfc}(x)$ for x large:

$$\mathrm{erfc}(x) = \frac{\mathrm{e}^{-x^2}}{x\sqrt{\pi}}\sum_{k=0}^{N}\frac{(-1)^k}{2^k}\,\frac{(2k-1)!!}{x^{2k}} + O\left(\frac{\mathrm{e}^{-x^2}}{x^{2N+3}}\right) \tag{38}$$

Note 1 *Equation (37) agrees with Proposition 1 - Equation (8) in the limit when $x \to 0$.*

3 Asymptotic expansions of the implied lognormal volatility

This section is intended for people like me who are not familiar with the notion of transseries. Otherwise, all the results below are supposed to be a simple consequence of the fact that $(\lambda, \ln(\lambda))$ form a transbase (See [6], Theorem 5.12).

We want now to express the time-variance $\theta^2 = \sigma_{LN}^2 T$ in terms of the time-value TV (resp. covered call CC) for $\theta \ll 1$ (resp. $\theta \gg 1$).

3.1 Asymptotic expansions of the implied lognormal volatility when $K \neq S$

Let us assume that $K \neq S$ i.e., $x \neq 0$. We need to invert Equations (3) and (8).

Our main result will be seen as a consequence of the following note.

Note 2 *Both Equations (3) and (8) are of the form:*

$$v^{\beta} \mathrm{e}^{-\frac{1}{v}} \left[\sum_{k=0}^{N} \alpha_k \, v^k \, + O\left(v^{N+1}\right) \right] = \mathrm{e}^{\gamma} \, \mathrm{e}^{-\frac{1}{\lambda}} \tag{39}$$

with:

- *(Case $\theta \ll 1$)* $v = \dfrac{2\,\theta^2}{x^2}$, $\beta = \dfrac{3}{2}$, $\alpha_k = \dfrac{(-1)^k}{2^k} \, . \, a_k \left(\dfrac{x^2}{8}\right)$, $\gamma = \ln\left(\dfrac{4\sqrt{\pi}\mathrm{e}^{-\frac{x}{2}}}{|x|}\right)$ *and* $\lambda = -\dfrac{1}{\ln(\frac{TV}{S})}$.

- *(Case $\theta \gg 1$)* $v = \dfrac{8}{\theta^2}$, $\beta = \dfrac{1}{2}$, $\alpha_k = \dfrac{(-1)^k}{2^k} \, . \, c_k \left(\dfrac{x^2}{8}\right)$, $\gamma = \ln\left(\sqrt{\pi}\,\mathrm{e}^{-\frac{x}{2}}\right)$ *and* $\lambda = -\dfrac{1}{\ln(\frac{CC}{S})}$.

We are going to invert (39) and thus to obtain an asymptotic expansion of v in terms of $\lambda^{\alpha} \ln(\lambda)^{\beta}$.

Lemma 1 *For any $(\alpha_k) \in \mathbb{R}^{\mathbb{N}}, \gamma \in \mathbb{R}$ and $N \in \mathbb{N}^*$, the asymptotics expansion of (39) for $0 < \lambda \ll 1$ and $\beta > 0$ is given by:*

$$v = \lambda - \beta\lambda^2 \ln\lambda + \gamma\lambda^2 + \beta^2\lambda^3 \ln^2(\lambda) + \left(\beta^2 - 2\beta\gamma\right) \lambda^3 \ln(\lambda) + \left(\gamma^2 - \beta\gamma - \alpha_1\right) \lambda^3 + o\left(\lambda^3\right) \tag{40}$$

Proof.

Order λ

We use the fact that if $f \sim g$ with $\lim f = 0^+$ or $+\infty$, then also $\ln f \sim \ln g$. Therefore, from

$$v^{\beta} \mathrm{e}^{-\frac{1}{v}} \sim \mathrm{e}^{\gamma} \, \mathrm{e}^{-\frac{1}{\lambda}} \tag{41}$$

and the fact that $\lim\limits_{\lambda \to 0} \mathrm{e}^{\gamma} \, \mathrm{e}^{-\frac{1}{\lambda}} = 0$, we get:

$$\beta \ln v - \frac{1}{v} \sim \gamma - \frac{1}{\lambda}$$

The function $g : x \mapsto \beta \ln(x) - \dfrac{1}{x}$ is non-decreasing and $\lim\limits_{x \to 0^+} g(x) = -\infty$. So, $\lim\limits_{\lambda \to 0} v = 0^+$. Moreover, since $\lim\limits_{v \to 0} v = 0$ and $\lim\limits_{v \to 0} v \ln v = 0$, we get $v \sim \lambda$.

Order $\lambda^2 \ln(\lambda)$

Let us define w by $v = \lambda(1 + w)$. Necessarily, $\lim w = 0$. Let us also denote by ε the function such that $\lim \varepsilon = 0$ and

$$v^{\beta} \mathrm{e}^{-\frac{1}{v}} (1 + \varepsilon) = \mathrm{e}^{\gamma} \, \mathrm{e}^{-\frac{1}{\lambda}}$$

We have:

$$\beta \ln v - \frac{1}{v} + \ln(1+\varepsilon) = \gamma - \frac{1}{\lambda}$$

So,

$$\frac{1}{v} - \frac{1}{\lambda} = \beta \ln v + \ln(1+\varepsilon) - \gamma$$

The right hand side of the last equality is clearly equivalent to $\beta \ln v$ when λ (and so also v) goes to 0. So,

$$-\frac{w}{v} \sim \beta \ln v$$

Thus,

$$w \sim -\beta v \ln v \sim -\beta \lambda \ln \lambda$$

So, we have proved:

$$v = \lambda - \beta \lambda^2 \ln \lambda + o\left(\lambda^2 \ln \lambda\right) \tag{42}$$

Order λ^2

By (41), we have:

$$\beta \ln v - \frac{1}{v} = \gamma - \frac{1}{\lambda} + o(1) \tag{43}$$

Set

$$v = \lambda\left(1 - \beta \lambda \ln(\lambda) + z\right) \tag{44}$$

with $z = o(\lambda \ln(\lambda))$. Then,

$$\ln(v) = \ln(\lambda) + o(1) \tag{45}$$

and

$$\begin{aligned} \frac{1}{v} &= \frac{1}{\lambda}\left[1 - \beta\lambda \ln(\lambda) + z\right]^{-1} \\ &= \frac{1}{\lambda}\left(1 + \beta \lambda \ln(\lambda) - z + o(\lambda)\right) \\ &= \frac{1}{\lambda} + \beta \ln(\lambda) - \frac{z}{\lambda} + o(1) \end{aligned} \tag{46}$$

Therefore, by (43),(45) and (46), we obtain:

$$\frac{z}{\lambda} = \gamma + o(1)$$

So,

$$z \sim \gamma \lambda \tag{47}$$

We have proved:

$$v = \lambda - \beta \lambda^2 \ln \lambda + \gamma \lambda^2 + o\left(\lambda^2\right) \tag{48}$$

Order $\lambda^3 \ln^2(\lambda)$

Let ξ be defined by

$$v = \lambda\left(1 - \beta \lambda \ln(\lambda) + \gamma \lambda + \xi\right) \tag{49}$$

Then, $\xi = o(\lambda)$ and

$$\begin{aligned} \ln(v) &= \ln(\lambda) + \ln\left(1 - \beta\lambda\ln(\lambda) + \gamma\lambda + \xi\right) \\ &= \ln(\lambda) + O\left(\lambda\ln(\lambda)\right) \\ &= \ln(\lambda) + o\left(\lambda\ln^2(\lambda)\right) \end{aligned} \tag{50}$$

On the other hand,

$$\begin{aligned} \frac{1}{v} &= \frac{1}{\lambda}\left[1 - \beta\lambda\ln(\lambda) + \gamma\lambda + \xi\right]^{-1} \\ &= \frac{1}{\lambda}\left[1 + \beta\lambda\ln(\lambda) - \gamma\lambda - \xi + \beta^2\lambda^2\ln^2(\lambda) + o\left(\lambda^2\ln^2(\lambda)\right)\right] \\ &= \frac{1}{\lambda} + \beta\ln(\lambda) - \gamma - \frac{\xi}{\lambda} + \beta^2\lambda\ln^2(\lambda) + o\left(\lambda\ln^2(\lambda)\right) \end{aligned} \tag{51}$$

With $N = 1$, Equation (39) says that

$$\beta\ln(v) - \frac{1}{v} + \ln\left(1 + \alpha_1 v + o(v)\right) = \gamma - \frac{1}{\lambda} \tag{52}$$

We have

$$\begin{aligned} \ln\left(1 + \alpha_1 v + o(v)\right) &\sim v \\ &\sim \lambda \\ &= o\left(\lambda\ln^2(\lambda)\right) \end{aligned}$$

So, by (50) and (51), we get:

$$\frac{\xi}{\lambda} - \beta^2\lambda\ln^2(\lambda) + o\left(\lambda\ln^2(\lambda)\right) = 0 \tag{53}$$

Therefore,

$$v = \lambda - \beta\lambda^2\ln\lambda + \gamma\lambda^2 + \beta^2\lambda^3\ln^2(\lambda) + o\left(\lambda^3\ln^2(\lambda)\right) \tag{54}$$

Order $\lambda^3\ln(\lambda)$

Set ϕ so that

$$v = \lambda\left(1 - \beta\lambda\ln(\lambda) + \gamma\lambda + \beta^2\lambda^2\ln^2(\lambda) + \phi\right) \tag{55}$$

with $\phi = o\left(\lambda^2\ln^2(\lambda)\right)$. We have:

$$\ln(v) = \ln(\lambda) - \beta\lambda\ln(\lambda) + o\left(\lambda\ln(\lambda)\right) \tag{56}$$

and

$$\begin{aligned} \frac{1}{v} &= \frac{1}{\lambda}\left(1 - \beta\lambda\ln(\lambda) + \gamma\lambda + \beta^2\lambda^2\ln^2(\lambda) + \phi\right)^{-1} \\ &= \frac{1}{\lambda}\left(1 + \beta\lambda\ln(\lambda) - \gamma\lambda - \beta^2\lambda^2\ln^2(\lambda) - \phi + \beta^2\lambda^2\ln^2(\lambda) - 2\beta\gamma\lambda^2\ln(\lambda) + o\left(\lambda^2\ln(\lambda)\right)\right) \\ &= \frac{1}{\lambda} + \beta\ln(\lambda) - \gamma - \frac{\phi}{\lambda} - 2\beta\gamma\lambda\ln(\lambda) + o\left(\lambda\ln(\lambda)\right). \end{aligned} \tag{57}$$

So,

$$\beta\ln(v)-\frac{1}{v}=\gamma-\frac{1}{\lambda}-\beta^2\lambda\ln(\lambda)+2\beta\gamma\lambda\ln(\lambda)+\frac{\phi}{\lambda}+o\left(\lambda\ln(\lambda)\right)$$

On the other hand, we have:

$$\begin{aligned}\ln\left(1+\alpha_1 v+o(v)\right) &\sim \alpha_1 v\\ &\sim \alpha_1\lambda\\ &= o\left(\lambda\ln(\lambda)\right)\end{aligned}\tag{58}$$

Thus, by (52), we deduce that

$$\frac{\phi}{\lambda}=\beta^2\lambda\ln(\lambda)-2\beta\gamma\lambda\ln(\lambda)+o\left(\lambda\ln(\lambda)\right)$$

and so,

$$\phi\sim\left(\beta^2-2\beta\gamma\right)\lambda^2\ln(\lambda)$$

Therefore,

$$v=\lambda-\beta\lambda^2\ln\lambda+\gamma\lambda^2+\beta^2\lambda^3\ln^2(\lambda)+\left(\beta^2-2\beta\gamma\right)\lambda^3\ln(\lambda)+o\left(\lambda^3\ln(\lambda)\right)\tag{59}$$

Order λ^3

Set $\psi=o\left(\lambda^2\ln(\lambda)\right)$ such that:

$$v=\lambda\left(1-\beta\lambda\ln(\lambda)+\gamma\lambda+\beta^2\lambda^2\ln^2(\lambda)+\left(\beta^2-2\beta\gamma\right)\lambda^2\ln(\lambda)+\psi\right).\tag{60}$$

Then,

$$\begin{aligned}\ln(v) &= \ln(\lambda)+\ln\left(1-\beta\lambda\ln(\lambda)+\gamma\lambda+\beta^2\lambda^2\ln^2(\lambda)+\left(\beta^2-2\beta\gamma\right)\lambda^2\ln(\lambda)+\psi\right)\\ &= \ln(\lambda)-\beta\lambda\ln(\lambda)+\gamma\lambda+o(\lambda) \qquad (61)\\ & \qquad (62)\end{aligned}$$

Also,

$$\begin{aligned}\frac{1}{v} &= \frac{1}{\lambda}\left(1-\beta\lambda\ln(\lambda)+\gamma\lambda+\beta^2\lambda^2\ln^2(\lambda)+\left(\beta^2-2\beta\gamma\right)\lambda^2\ln(\lambda)+\psi\right)^{-1}\\ &= \frac{1}{\lambda}\left(1+\beta\lambda\ln(\lambda)-\gamma\lambda-\beta^2\lambda^2\ln^2(\lambda)-\left(\beta^2-2\beta\gamma\right)\lambda^2\ln(\lambda)-\psi\right)\\ &+ \frac{1}{\lambda}\left(\beta^2\lambda^2\ln^2(\lambda)-2\beta\gamma\lambda^2\ln(\lambda)+\gamma^2\lambda^2+o(\lambda^2)\right)\\ &= \frac{1}{\lambda}+\beta\ln(\lambda)-\gamma-\beta^2\lambda\ln(\lambda)+\gamma^2\lambda-\frac{\psi}{\lambda}+o(\lambda)\end{aligned}$$

and

$$\begin{aligned}\ln(1+\alpha_1 v+o(v)) &= \alpha_1 v+o(v)\\ &= \alpha_1\lambda+o(\lambda)\end{aligned}\tag{63}$$

Therefore,

$$\beta\ln(v)-\frac{1}{v}+\ln(1+\alpha_1 v+o(v))-\gamma+\frac{1}{\lambda}=\beta\gamma\lambda-\gamma^2\lambda+\frac{\psi}{\lambda}+\alpha_1\lambda+o(\lambda)$$

By (52), the left hand side of this equation is 0. So,

$$\frac{\psi}{\lambda} = \left(\gamma^2 - \beta\gamma - \alpha_1\right)\lambda + o(\lambda)$$

and

$$\psi = \left(\gamma^2 - \beta\gamma - \alpha_1\right)\lambda^2 + o(\lambda^2)$$

This put an end to Lemma 1.

By induction on m and n, we can also prove the following generalization of Lemma 1.

Proposition 4 *There are $a_{i,j}$ defined for $(i,j)^2 \in \mathbb{N}$ and $j < i$ such that for any $(m,n) \in \mathbb{N}^2$, with $n < m$, we have:*

$$v = v_{m,n} + o\left(\lambda^m \ln^n(\lambda)\right) \tag{64}$$

with

$$v_{m,n} := \sum_{i=1}^{m}\sum_{j=n}^{m-1} a_{i,j}\,\lambda^i \ln^j(\lambda) \tag{65}$$

- *We have $\lambda \succ \lambda^2\ln(\lambda) \succ \lambda^2 \succ \lambda^3\ln^2(\lambda) \succ \lambda^3\ln(\lambda) \succ \lambda^3 \succ \lambda^4\ln^3(\lambda) \succ ....$ The symbol $\succ$ is defined by $f \succ g$ if and only if $g = o(f)$ in a neighborhood of 0.*
- *In this sequence, $\lambda^i\ln^j(\lambda)$ is in position $\pi_{i,j} := \dfrac{i(i+1)}{2} - j$.*
- *for any $k \in \mathbb{N}$, there is a unique $(i,j) \in \mathbb{N}$ with $j < i$ such that $k = \pi_{i,j}$.*
- *we set $v_k := v_{i,j}$ with $k = \pi_{i,j}$.*

For $m \geq 3$, the coeficient $a_{m,n}$ can be obtained by induction by the following way:

- *We expand $\ln\left(\dfrac{v_{\pi_{m,n}-1}}{\lambda}\right)$, $\dfrac{1}{v_{\pi_{m,n}-1}}$ and $\ln\left(\displaystyle\sum_{k=0}^{m-2}\alpha_k v^k\right)$ and we keep the terms in $\lambda^{m-2}\ln^n(\lambda)$.*
- *We note those terms $A_{m,n}, B_{m,n}$ and $C_{m,n}$ respectively.*
- *Then, $a_{m,n} = B_{m,n} - \dfrac{3}{2}A_{m,n} - C_{m,n}$.*

As an application of Note 2 and Lemma 1, we get:

Proposition 5 *(Case 1: short expiry). Let us denote by $TV := C(T,K) - (S-K)_+$ the time-value of a European call option, σ_{LN} its implied lognormal volatility and T the maturity of the option. Set $\lambda := -\dfrac{1}{\ln(\frac{TV}{S})}$, $\gamma := \ln\left(\dfrac{4\sqrt{\pi}\mathrm{e}^{-\frac{x}{2}}}{|x|}\right)$ and $\alpha_1 = -\dfrac{3}{2} - \dfrac{x^2}{16}$ with $x = \ln(\frac{K}{S})$. Then, when $T \to 0$, we have the following expansion for the time-variance of the call option: $\sigma_{LN}^2 T = \dfrac{x^2}{2}v$ with*

$$v = \lambda - \frac{3}{2}\lambda^2\ln\lambda + \gamma\lambda^2 + \frac{9}{4}\lambda^3\ln^2(\lambda) + \left(\frac{9}{4} - 3\gamma\right)\lambda^3\ln(\lambda) + \left(\gamma^2 - \frac{3}{2}\gamma - \alpha_1\right)\lambda^3 + o\left(\lambda^3\right)$$

(Case 2: large expiry). Let us denote by $CC := S - C(T,K)$ the covered call of a European call option, σ_{LN} its implied lognormal volatility and T the maturity of the option. Set $\lambda := -\frac{1}{\ln(\frac{CC}{S})}$, $\gamma := \ln\left(\sqrt{\pi}\,\mathrm{e}^{-\frac{x}{2}}\right)$ and $\alpha_1 = -\frac{1}{2} - \frac{x^2}{16}$ with $x = \ln(\frac{K}{S})$. Then, when $T \to +\infty$, we have the following expansion for the time-variance of the call option:

$$\sigma_{LN}^2 T = \frac{8}{\lambda}\left[1 + \frac{1}{2}\lambda\ln(\lambda) - \gamma\lambda - \frac{1}{4}\lambda^2\ln(\lambda) + \left(\frac{\gamma}{2} + \alpha_1\right)\lambda^2 + o\left(\lambda^2\right)\right] \tag{66}$$

In particular, when $T \to +\infty$, $\quad \sigma_{LN} \sim 2\sqrt{-\frac{2\ln\left(\frac{CC}{S}\right)}{T}}$.

Equation (66) is a generalization of [5].

Proof. Case 1 is just an application of Note 2 and Lemma 1. Case 2 follows from the expansion of v^{-1}. Indeed, by (40), we have:

$$\frac{1}{v} = \frac{1}{\lambda}\left[1 + \beta\lambda\ln(\lambda) - \gamma\lambda - \beta^2\lambda^2\ln(\lambda) + (\beta\gamma + \alpha_1)\lambda^2 + o\left(\lambda^2\right)\right] \tag{67}$$

Therefore using Note 2 - Case 2,

$$\frac{\sigma_{LN}^2 T}{8} = \frac{1}{\lambda}\left[1 + \frac{1}{2}\lambda\ln(\lambda) - \gamma\lambda - \frac{1}{4}\lambda^2\ln(\lambda) + (\frac{\gamma}{2} + \alpha_1)\lambda^2 + o\left(\lambda^2\right)\right]$$

Hence, we get the result.

Note 3 *The case $x \to +\infty$ and θ fixed (Case 2 of Proposition 1) can be treated exactly in the same way. It is more or less exactly the same as the case x fixed and $\theta \to 0$ except that β is now equal to 1, γ has to be replaced by $\ln\left(2\sqrt{2\pi}\,\frac{\mathrm{e}^{-\frac{x}{2}}}{\theta}\right)$, and α_k ($k \in \mathbb{N}$) has to be replaced by $\frac{(-1)^k}{2^k}\, b_k\left(\frac{\theta^2}{4}\right)$ with b_k given in (7). Therefore, $\alpha_1 = -\frac{3}{2} + \frac{x^2}{8}$ and the formula for the implied lognormal volatility is $\sigma_{LN}^2 = \frac{x^2}{2}\, v$ with*

$$v = \lambda - \lambda^2\,\ln\lambda + \gamma\lambda^2 + \lambda^3\,\ln^2(\lambda) + (1 - 2\gamma)\,\lambda^3\ln(\lambda) + \left(\gamma^2 - \gamma - \alpha_1\right)\lambda^3 + o\left(\lambda^3\right) \tag{68}$$

3.2 Implied lognormal volatility at the money

It turns out that at the money, there is a closed form formula for implied lognormal volatility in terms of call price. No asumption on T is made.

Proposition 6 *At the money, implied lognormal volatility σ_{LN} can be obtained as a power series in call price C according to the formula:*

$$\sigma_{LN} = \sqrt{\frac{2\pi}{T}}\,\frac{C}{S}\sum_{k=0}^{\infty}\frac{\pi^k\eta_k}{4^k(2k+1)}\left(\frac{C}{S}\right)^{2k} \tag{69}$$

with η_k given by induction:

$$\eta_k = \sum_{j=0}^{k}\frac{\eta_j\,\eta_{k-1-j}}{(j+1)(2j+1)} \tag{70}$$

Proof. We have the well known expansion of erf^{-1} (see for instance [1]):

$$\mathrm{erf}^{-1}(x) = \sum_{k=0}^{\infty} \frac{\eta_k}{2k+1} \left(\frac{\sqrt{\pi}}{2} x \right)^{2k+1} \tag{71}$$

$$\eta_k = \sum_{j=0}^{k} \frac{\eta_j \, \eta_{k-1-j}}{(j+1)(2j+1)} \tag{72}$$

So, by (34),

$$\begin{aligned} \theta &= 2\sqrt{2} \, \frac{\sqrt{\pi}}{2} \, \frac{C}{S} \sum_{k=0}^{\infty} \frac{\eta_k}{(2k+1)} \left(\frac{\sqrt{\pi}}{2} \right)^{2k} \left(\frac{C}{S} \right)^{2k} \\ &= \sqrt{2\pi} \, \frac{C}{S} \sum_{k=0}^{\infty} \frac{\pi^k \eta_k}{4^k (2k+1)} \left(\frac{C}{S} \right)^{2k} \end{aligned} \tag{73}$$

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