Four Things You Might Not Know About the Black-Scholes Formula

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The Black-Scholes formula is the mother of all option pricing formulas. It states that under perfect market conditions and geometric Brownian motion dynamics, the only arbitrage-free time-t price of a strike-K expiry-T call option is

$$Call(t) = BS^{call}(S(t),\ T-t,\ K,\ r,\ \sigma)$$

where $S(t)$ is the time-t price of a dividend-free stock,[1] r is the risk-free rate, σ is volatility (i.e., the standard deviation of appropriately time-scaled returns), and the function BS^{call} is given by

$$BS^{call}(S,\tau,K,r,\sigma) = S\,N(d_1(S,\tau,K,r,\sigma))$$
$$-e^{-r\tau}K\,N(d_2(S,\tau,K,r,\sigma))$$

where N denotes the standard normal distribution function, $\tau = T - t$, and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

The Black-Scholes formula can be derived in a number of ways. Andreasen, Jensen, and Poulsen [1998] is an account of some of them; Derman and Taleb [2005] is a recent (although debatable, see Ruffino and Treussard [2006]) addition.

In this article, I discuss some less well-known results related to the Black-Scholes formula. You may see them just as "cute," "quaint," or "a nice exercise," but they go deeper. As I briefly outline, they are special cases of more general results or techniques. Or in a poetic sense, they are the shadows cast from higher dimensions onto the walls of the Black-Scholes cave.

DERIVING DELTA— CORRECTLY— WITHOUT LENGTHY CALCULATIONS

A central quantity in hedging and risk management is the call (or any other) option's sensitivity to changes in the stock-price—its delta, which is defined as

$$\Delta = \frac{\partial BS^{call}}{\partial S}$$

A tempting way to a show this is to ignore (forget) that S enters inside the $N(...)$-expressions, which makes the following differentiation very easy:

$$\Delta = N(d_1)$$

Rather unpedagogically, this happens to be the correct result. To derive it properly you must use the chain rule when differentiating. This gives two extra terms that cancel after tedious calculations.

A simpler derivation that does not appear to be well-known is that a function f (defined on a cone-shaped domain of $\Re^n$) is said to be homogenous (of degree one) if $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \Re_+$ and all x (in f's domain). Euler's Theorem (used predominantly in microeconomics) states that a differentiable function f is homogenous if, and only if, it has the form $f(x) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$. Now observe that the Black-Scholes call price is homogenous in stock price and strike. Euler's Theorem tells us that the term which "multiplies S" in the formula is indeed the partial derivative with respect to S—the delta.

This homogeneity property (known in the financial engineering literature as the *sticky moneyness regime*) holds not just in the Black-Scholes model, but as discussed in Joshi [2003, chapter 15] in a more general class where the return distribution is independent of the current stock-price level. That involves affine jump-diffusions as well as some infinite intensity Levy-driven processes. Before we get carried away, Lee [2004] shows that although call prices in these models can be written such that they look a lot like the Black-Scholes formula (from which delta is then recognized), that is far from the best representation for numerical calculations.

Not all models are homogeneous. The Bachelier model (where S is an arithmetic Brownian motion), constant elasticity-of-variance model, SABR stochastic volatility model as well as Dupire-Derman-type local volatility models are inhomogeneous. Devising an empirical methodology that is powerful enough for testing (i.e., for rejecting) homogeneity remains an open problem.[2]

PUT–CALL DUALITY

If you plug in $-\sigma$ into the call-price formula, the result is a negative put price,

$$BS^{call}(S(t), \tau, K, r, -\sigma) = S\,N(-d_1) - e^{-r\tau}\,K\,N(-d_2)$$
$$= S(1 - N(d_1)) - e^{-r\tau}\,K(1 - N(d_2))$$
$$= S - e^{-r\tau}\,K - BS^{call}(S(t), \tau, K, r, \sigma)$$
$$= -BS^{put}(S(t), \tau, K, r, \sigma)$$

where the first equality uses symmetry of the normal distribution and the third employs put–call parity. In the Black-Scholes model, this put–call duality is little more than quaint, but it is related to time reversal which may, among other things, play a useful role in the efficient calibration of local volatility models (see Andreasen, Jensen,

and Poulsen [1998, section 7] and Peskir and Shiryaev [2001]).

A SIMULATION ENGINE

The binomial model is often called the workhorse of finance. In that vein, the pseudo-code below may be termed a *simulation engine*. It mimics the behavior of a discrete delta-hedger.

```
V=BScall(S,T,K,r,sigma,greek=price)
                        # initial investment
a=BScall(S,T,K,r,sigma,greek=delta)
                  # stock position = delta
b=V-a*S           # rest in bank; self-fin. Cond.
Loop over j = 1 to Npaths
     Loop over i=1 to Nhedgepoints
          eps=random.normal()
                  # simulate outcome of N(0,1)
          S=S*exp((mu-0.5*sigma^2)*dt+*sqrt (dt)
                                          *eps)
          V=a*S+b*exp(r*dt)
          a=BScall(S,T-i*dt,K,r,sigma,greek=delta)
          b=V-a*S
     next i
     hedge(j) = V
     error(j) = max(S-K,0) - V
next j
```

The first "Eureka!" moment is realizing that the engine works. Exhibit 1 clearly shows that as the hedge frequency increases, the standard deviation of the hedge error (and hence the hedge error itself) goes to zero. The log/log-scaling and the fitted line of the form *constant* -0.5 log(#hedge points) reveal that the order at which this happens is as (one over) the square root of the hedge frequency.[3]

Digressing slightly, let me say that I have found the simulation engine to be very useful in a teaching context. Among the points that can readily be illustrated are that

- hedge error goes to zero irrespective of the drift in the simulations, but not if you use the wrong volatility (the layman's Girsanov).
- the discounted-value process of any self-financing trading strategy is a martingale under the risk-neutral probability measure. This means that if you simulate with $\mu = r$, the average discounted payoff equals the initial investment. This is very good for detecting coding errors. (Try permuting the innermost lines in the loops.) This is also a good place to have students ponder the $\sigma^2/2$ Ito term.

E X H I B I T 1
Standard Deviation in Pure Delta-Hedging

Standard deviation of the hedge error (as a percentage of the initial option price) for discrete delta-hedging in the Black–Scholes model. The option being hedged is a 1-year call with strike = 1.15*spot. The risk-free rate is 5%, the drift of the stock is 10%, and its volatility is 20%.
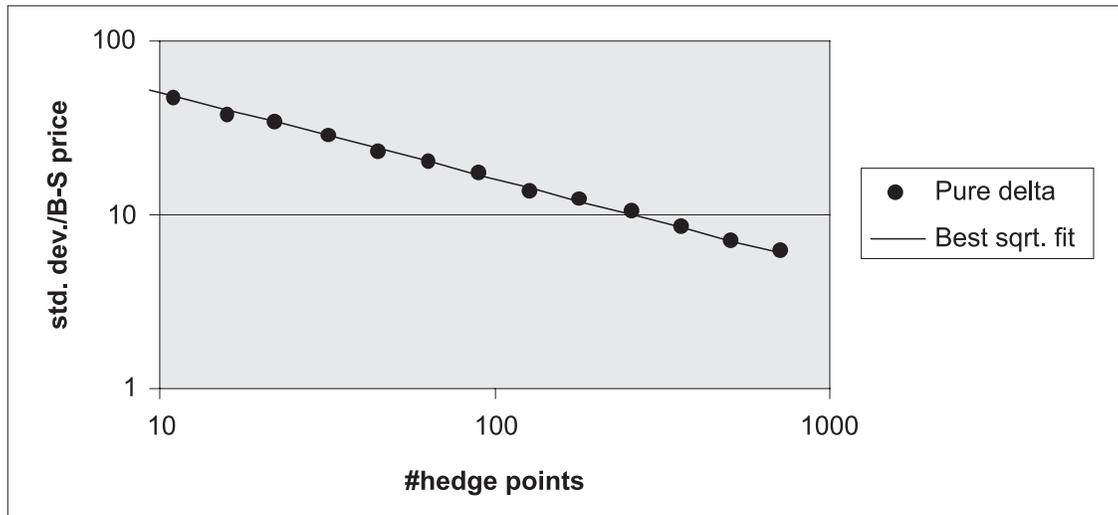
• digital options are hard to hedge, so try different contracts (see Gobet and Teman [2001] in which this is specifically addressed).

As discussed in Björk [2004, chapter 9], to discretely hedge an out-of-the-money option (think of this as illiquid; strike K1), you may want include a closer-to-the-money option (think liquid; strike K0) to make your portfolio gamma-neutral. This can be achieved by the following adjustment to the code:

```
V=a*S+b*exp(r*dt)+c*BScall(S,T-i*dt, K0,...,greek
                                        =price)
c=BScall(S,T-i*dt,K1,...,greek=gamma)/BScall
                (S,T-i*dt, K0,...,greek=gamma)
a=BScall(S,T-i*dt,K1,...,greek=delta)-c*BScall
                (S,T-i*dt,K0,...,greek=delta)
b=V-a*S-c* BScall(S,T-i*dt,K0,..., greek=price)
```
(1)

The effects are shown by the lower curve in Exhibit 2. We see a significant improvement in the hedge performance in that the standard deviation of hedge errors is reduced by a factor of about 3. The fact that a second-order correction (as gamma-hedging can be thought of) improves the approximation is perfectly reasonable.

A trader might use exactly the same argument to make his portfolio vega-neutral (i.e., it is value insensitive to changes in volatility). This is achieved by changing line (1) to

```
c=BS(S,T-i*dt,K0,...,greek=vega)/BS(S,T-i*dt,K0,
                        ..., greek=vega)
```
(2)

Logically, this is nonsense. A key assumption in the Black–Scholes model is that volatility is constant. The trader may persist, however, saying that his experience shows that it improves hedge performance—and it does. To see why, recall that in the Black–Scholes model

$$vega := \frac{\partial BS^{call}}{\partial \sigma} = S\phi(d_1)\sqrt{\tau}$$

$$gamma := \frac{\partial^2 BS^{call}}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{\tau}}$$

so the ratios of vegas and gammas are the same (i.e., vega- and gamma-hedges from lines (1) and (2) are the same).

In stochastic volatility models, vega-dependent hedge strategies are not just an improvement, they are a necessity (see Joshi [2003] or Cont and da Fonseca [2002]). Furthermore, Ewald, Poulsen, and Schenk-Hoppe [2006] shows that for such hedges to be effective it is important

E X H I B I T   2
Standard Deviation in Delta- and Gamma-Hedging

Standard deviation of hedge errors for delta- and delta/gamma-hedging. The parameters are the same as in Exhibit 1. The gamma-hedge uses a forward at-the-money expiry-1call option.
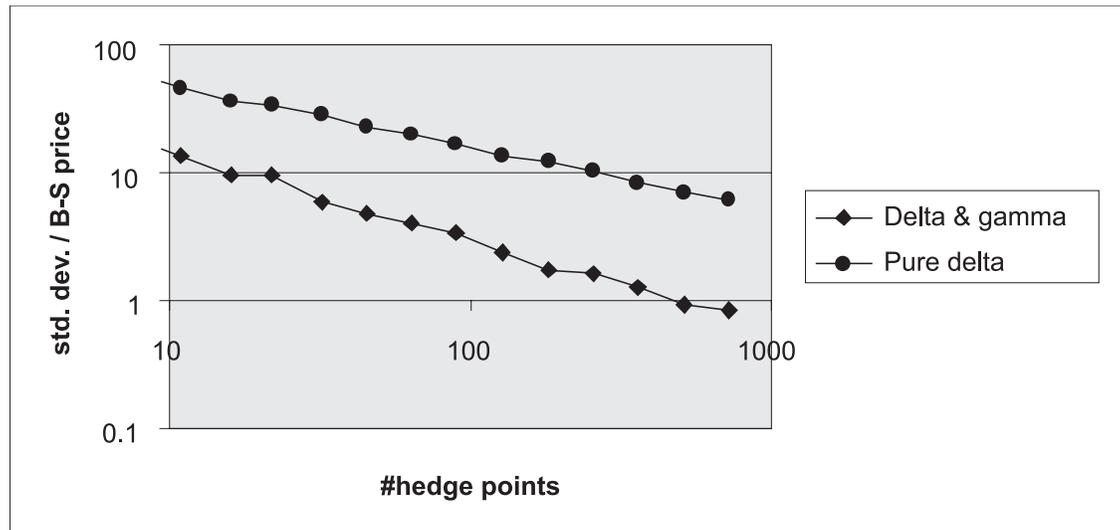
to use the delta, and especially vega, from a genuine stochastic volatility model. This can be interpreted as a Lucas critique in that sensitivities from a static model can be misleading in a dynamic model.

BEWARE OF GREEKS

The last of the "four things" mentioned in the article's title is a caveat—or a confession. I cheated when I produced Exhibit 2 which depicts the benefits of gamma-hedging over plain delta-hedging. If you use the equation in line (1) directly, you get a picture as shown in Exhibit 3. Nice at first, but if you hedge more often than about every other week, things go violently wrong. Daily hedging has a standard deviation on the order of magnitude of the number of atoms in the universe. The instability comes about because gamma goes to zero extremely quickly—exponentially squared fast—when the strike moves away from the spot. The closer we are to expiry, the worse things are. This means that even for strikes that are reasonably close, the denominator in line (1) can be much, much smaller than the numerator, and thus the option position in the hedge, c, becomes unreasonably large. A simple regularization is to truncate c at some level. This works—truncation at 10 was what produced Exhibit 2.

Another solution is always to use the forward at-the-money option or the one with the highest (Black-Scholes model) gamma to hedge with. Note that this means liquidating your entire hedge option portfolio each trading day. Regularization is investigated in a more general hedging-with-options context in Nalholm and Poulsen [2006] in which the benefits of singular value decomposition are demonstrated.

CONCLUSION

Although I have described four cases for which general methods have interesting consequences in the Black-Scholes model, I would be surprised if there are not more.

ENDNOTES

[1]A *no-dividends* assumption is not always without loss of generality; it may change "standard results." For instance, with a positive dividend yield call prices may decrease at long expiries. However, a dividend yield does not alter the results I present, so to ease the exposition, it has been left out.

[2]For instance, Cont and da Fonseca [2002] impose homogeneity from the outset in their empirical analysis.

[3]This has been known since Boyle and Emanuel [1980] and is thus not one of the "four things" mentioned in the title of this article.

E X H I B I T 3
Standard Deviation in Unregularized Gamma-Hedging

Hedge error standard deviation in gamma-hedging fixed-strike hedge option and no regularization. Note the units on the *y*-axis.
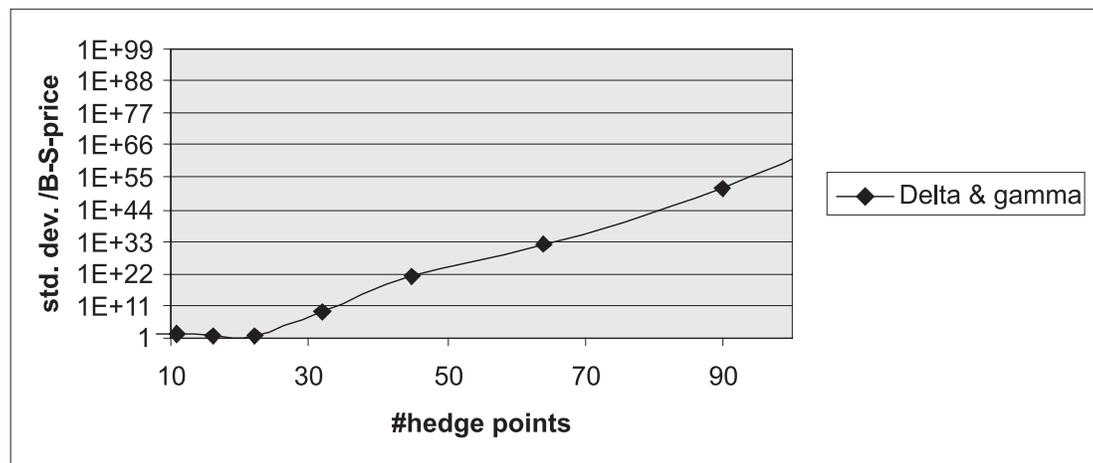
REFERENCES

Andreasen, Jesper, Bjarke Jensen, and Rolf Poulsen. "Eight Valuation Methods in Financial Mathematics: The Black-Scholes Formula as an Example." *Mathematical Scientist*, 23 (1998), pp. 18–40.

Björk, Thomas. *Arbitrage Theory in Continuous Time*. Oxford, England: Oxford University Press, 2004.

Boyle, Phelim P., and D. Emanuel "Discretely Adjusted Option Hedges." *Journal of Financial Economics*, 8 (1980), pp. 259–282.

Cont, Rama, and Jose da Fonseca. "Dynamics of Implied Volatility Surfaces." *Quantitative Finance*, 2 (2002), pp. 45–60.

Derman, Emmanuel, and Nassim Taleb. "The Illusions of Dynamic Replication." *Quantitative Finance*, 5 (2005), pp. 323–326.

Ewald, Christian-Oliver, Rolf Poulsen, and Klaus Reiner Schenk-Hoppe. "Stochastic Volatility: Risk-Minimization and Model Risk." Working paper, Leeds University, 2006.

Gobet, E., and E. Teman. "Discrete Time Hedging Errors for Options with Irregular Payoffs." *Finance and Stochastics*, 5 (2001), pp. 357–367.

Joshi, Mark. *Theory and Practice of Mathematical Finance*. Cambridge, England: Cambridge University Press, 2003.

Lee, Roger. "Option Pricing by Transform Methods: Extensions, Unification, and Error Control." *Journal of Computational Finance*, 7 (2004), pp. 51–86.

Nalholm, Morten, and Rolf Poulsen. "Static Hedging of Barrier Options under General Asset Dynamics: Unification and Application." *Journal of Derivatives*, 13 (2006), pp. 46–60.

Peskir, Goran, and Albert Shiryaev. "A Note on Put-Call Parity and Put-Call Duality." *Theory of Probability and Its Applications*, 46 (2001), pp. 167–169.

Ruffino, Doriana, and Jonathan Treussard. "Derman and Taleb's 'The Illusions of Dynamic Replication': A Comment." *Quantitative Finance*, 6 (2006), pp. 365–367.